

# DIMENSIONS OF $\ell^p$ -COHOMOLOGY GROUPS

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ABSTRACT. Let  $G$  be an infinite discrete group of type  $\text{FP}_\infty$  and let  $1 < p \in \mathbb{R}$ . We prove that the  $\ell^p$ -homology and cohomology groups of  $G$  are either 0 or infinite dimensional. We also show that the cardinality of the  $p$ -harmonic boundary of a finitely generated group is either 0, 1, or  $\infty$ .

## 1. INTRODUCTION

Let  $G$  be a discrete group and for  $1 \leq p \in \mathbb{R}$ , let

$$\ell^p(G) = \left\{ \sum_{x \in G} a_x x \mid a_x \in \mathbb{C} \text{ and } \sum_{x \in G} |a_x|^p < \infty \right\}.$$

This is a complex Banach space with respect to the norm  $\|f\|_p := \left( \sum_{x \in G} |a_x|^p \right)^{1/p}$ . Also  $G$  acts on the left of  $\ell^p(G)$  according to the rule  $g \sum_{x \in G} a_x x = \sum_{x \in G} a_x gx$ , and similarly on the right according to the rule  $\left( \sum_{x \in G} a_x x \right) g = \sum_{x \in G} a_x xg$ . These actions make  $\ell^p(G)$  into a  $\mathbb{C}G$ -bimodule. Suppose we are given a free resolution of the trivial  $\mathbb{C}G$ -module  $\mathbb{C}$  with free right  $\mathbb{C}G$ -modules:

$$(1.1) \quad \cdots \longrightarrow \mathbb{C}G^{e_{n+1}} \xrightarrow{d_n} \mathbb{C}G^{e_n} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathbb{C}G^{e_1} \xrightarrow{d_0} \mathbb{C}G \longrightarrow \mathbb{C} \longrightarrow 0.$$

Let

$$(1.2) \quad d_n^*: \text{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_n}, \ell^p(G)) \longrightarrow \text{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_{n+1}}, \ell^p(G)),$$

$$(1.3) \quad d_n^*: \mathbb{C}G^{e_{n+1}} \otimes_{\mathbb{C}G} \ell^p(G) \longrightarrow \mathbb{C}G^{e_n} \otimes_{\mathbb{C}G} \ell^p(G)$$

be the maps induced by  $d_n$ ; for convenience we write  $d_{-1}^* = d_0^* = 0$ . Then one has the usual (unreduced) cohomology and homology groups

$$H^n(G, \ell^p(G)) = \ker d_n^* / \text{im } d_{n-1}^*,$$

$$H_n(G, \ell^p(G)) = \ker d_n^* / \text{im } d_n^*.$$

We will be interested in the case  $e_n < \infty$  for all  $n$ , whence  $\text{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_n}, \ell^p(G)) \cong \ell^p(G)^{e_n}$  as left  $\mathbb{C}G$ -modules and  $\mathbb{C}G^{e_n} \otimes_{\mathbb{C}G} \ell^p(G) \cong \ell^p(G)^{e_n}$  as right  $\mathbb{C}G$ -modules. (Recall that  $\text{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_n}, \ell^p(G))$  consists of right  $\mathbb{C}G$ -maps  $\theta: \mathbb{C}G^{e_n} \rightarrow \ell^p(G)$  with left  $G$ -action defined by  $(g\theta)(\alpha) = g(\theta\alpha)$  for  $\alpha \in \mathbb{C}G^{e_n}$ . Also  $\mathbb{C}G^{e_{n+1}} \otimes_{\mathbb{C}G} \ell^p(G)$  is the tensor product of the right  $\mathbb{C}G$ -module  $\mathbb{C}G^{e_{n+1}}$  with the left  $\mathbb{C}G$ -module  $\ell^p(G)$  with right  $G$ -action defined by  $(\alpha \otimes u)g = \alpha \otimes ug$ .)

Bekka and Valette [2, Corollary 8] proved that if  $G$  is a finitely generated group, then  $H^1(G, \ell^2(G))$  is either zero or infinite dimensional. The motivation behind

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this paper is to see if this result holds for arbitrary  $1 < p \in \mathbb{R}$ . It turns out that this result remains true not only for  $1 < p \in \mathbb{R}$ , but also for the other homology and cohomology groups. Recall that  $G$  is of type  $\text{FP}_n$  over  $\mathbb{C}$  if there exists a resolution (1.1) which has  $e_d$  finite for all  $d \leq n$ , and  $G$  is of type  $\text{FP}_\infty$  if it is of type  $\text{FP}_n$  for all  $n \in \mathbb{N}$ . Furthermore  $G$  is of type  $\text{FP}_1$  over  $\mathbb{C}$  if and only if  $G$  is finitely generated. We shall prove

**Theorem 1.4.** *Let  $d, n$  be non-negative integers and let  $G$  be an infinite group of type  $\text{FP}_n$  over  $\mathbb{C}$ . Let  $1 < p \in \mathbb{R}$ . Then*

- (i)  $H^d(G, \ell^p(G))$  is either 0 or has infinite  $\mathbb{C}$ -dimension for all  $d \leq n$ .
- (ii)  $H_d(G, \ell^p(G))$  is either 0 or has infinite  $\mathbb{C}$ -dimension for all  $d \leq n$ .

Theorem 1.4 immediately yields

**Corollary.** *Let  $G$  be an infinite group of type  $\text{FP}_\infty$  over  $\mathbb{C}$  and let  $d$  be a non-negative integer. Let  $1 < p \in \mathbb{R}$ . Then*

- (i)  $H^d(G, \ell^p(G))$  is either 0 or has infinite  $\mathbb{C}$ -dimension.
- (ii)  $H_d(G, \ell^p(G))$  is either 0 or has infinite  $\mathbb{C}$ -dimension.

We deduce Theorem 1.4 from our main theorem:

**Theorem 1.5.** *Let  $G$  be an infinite group, let  $m$  be a non-negative integer, and let  $A \subseteq B$  be closed left  $G$ -invariant subspaces of  $\ell^p(G)^m$ . Then either  $A = B$  or  $B/A$  has infinite dimension over  $\mathbb{C}$ .*

Of course, Theorem 1.5 remains true if we replace “left” with “right”.

The layout of this paper is as follows. In Section 2 we give some definitions and recall some well-known results. In Section 3 we prove Theorems 1.4 and 1.5. In Section 4, we shall use Theorem 1.5 to obtain a result concerning the cardinality of the  $p$ -harmonic boundary of a finitely generated group and to prove a result about translation invariant functionals on a certain function space of functions on a finitely generated group.

## 2. PRELIMINARIES

We denote the positive integers by  $\mathbb{N}$ . Let  $1 \leq p \in \mathbb{R}$ . Then for  $\alpha = \sum_{g \in G} \alpha_g g \in \ell^1(G)$  and  $\beta = \sum_{g \in G} \beta_g g \in \ell^p(G)$ , we define convolution by

$$\alpha\beta = \sum_{g, h \in G} \alpha_g \beta_h gh = \sum_{g \in G} \left( \sum_{x \in G} \alpha_{gx^{-1}} \beta_x \right) g \in \ell^p(G).$$

Young’s inequality [5, 32D] tells us that

$$(2.1) \quad \|\alpha\beta\|_p \leq \|\alpha\|_1 \|\beta\|_p.$$

Thus in particular  $\ell^1(G)$  is a ring with multiplication being convolution. Let  $m$  be a non-negative integer. While different norms can be defined on a finite direct sum of normed spaces, they are all equivalent [6, §1.8]. The most natural and consistent choice for  $\ell^p(G)^m$  is using the  $p$ -norm:

$$\|(u_1, \dots, u_m)\|_p = \left( \sum_{k=1}^m \|u_k\|_p^p \right)^{1/p} \text{ for all } (u_1, \dots, u_m) \in \ell^p(G)^m.$$

The inequality (2.1) still holds for  $u \in \ell^1(G)$  and  $v = (v_1, \dots, v_m) \in \ell^p(G)^m$  with convolution defined componentwise, because

$$\begin{aligned}
 \|uv\|_p^p &= \|u(v_1, \dots, v_m)\|_p^p = \sum_{k=1}^m \|uv_k\|_p^p \leq \sum_{k=1}^m (\|u\|_1 \|v_k\|_p)^p \\
 (2.2) \qquad &= \sum_{k=1}^m \|u\|_1^p \|v_k\|_p^p = \|u\|_1^p \sum_{k=1}^m \|v_k\|_p^p = \|u\|_1^p \|v\|_p^p.
 \end{aligned}$$

Similarly for  $u \in \ell^p(G)$  and  $v = (v_1, \dots, v_m) \in \ell^1(G)^m$ , we have

$$(2.3) \qquad \|uv\|_p^p \leq \|u\|_p^p \|v\|_1^p.$$

Note that (2.2) tells us that  $\ell^p(G)^m$  is a left  $\ell^1(G)$ -module, and that closed left  $G$ -invariant subspaces of  $\ell^p(G)^m$  are left  $\ell^1(G)$ -submodules.

We shall also need the following two well-known results.

**Lemma 2.4.** *Suppose  $n$  is a positive integer and for  $1 \leq k \leq n$  we have bounded linear operators  $T_k: B \rightarrow B$  on a normed space  $B$  such that the range  $T_k(B)$  is dense in  $B$  for each  $T_k$ . Then the range of  $T_1 \cdots T_n$  is also dense in  $B$ .*

*Proof.* First we prove the claim for  $n = 2$ . Let  $\epsilon > 0$  and  $b \in B$ . There exists  $b_1 \in B$  such that  $\|b - T_1 b_1\| < \epsilon/2$ , and then there exists  $b_2 \in B$  such that  $\|b_1 - T_2 b_2\| \leq \frac{\epsilon}{2\|T_1\|}$ . Thus

$$\begin{aligned}
 \|b - T_1 T_2 b_2\| &\leq \|b - T_1 b_1\| + \|T_1 b_1 - T_1 T_2 b_2\| \\
 &\leq \|b - T_1 b_1\| + \|T_1\| \|b_1 - T_2 b_2\| \\
 &< \frac{\epsilon}{2} + \|T_1\| \cdot \frac{\epsilon}{2\|T_1\|} = \epsilon.
 \end{aligned}$$

The lemma now follows by induction on  $n$ . □

**Lemma 2.5.** *Let  $T: A \rightarrow B$  be a bounded linear operator between the Banach spaces  $A$  and  $B$ . If  $T(A)$  has finite codimension in  $B$ , then  $T(A)$  is closed in  $B$ .*

*Proof.* See [1, p. 95, Exercise (1), §3.4] □

### 3. PROOF OF THE MAIN THEOREMS

The critical case in the proof of Theorem 1.5 is when  $G$  is infinite cyclic, and the reader will understand most of the proof by studying this special situation. To prove the result in general, we have had to repeat some arguments almost verbatim several times. However we have chosen to give full details over brevity and clarity.

*Proof of Theorem 1.5.* We will assume that  $B/A$  is finite dimensional and will prove that  $A = B$ . First suppose  $G$  has an element  $g$  of infinite order. Write  $H = \langle g \rangle$ . Note that for  $\alpha \in \ell^1(G)$  and  $\beta \in \ell^p(G)^m$ , we have  $\|\alpha\beta\|_p \leq \|\alpha\|_1 \|\beta\|_p$  by (2.2), thus in particular  $\ell^p(G)^m$  is a left  $\ell^1(H)$ -module, and  $A$  and  $B$  are left  $\ell^1(H)$ -submodules. The action of  $g$  on the finite dimensional vector space  $B/A$  has a minimal polynomial, i.e. there exists  $F(x) \in \mathbb{C}[x]$  such that  $F(g) = 0$  on  $B/A$ , and therefore  $F(g)b \in A$  for all  $b \in B$ . Factor  $F(x)$  into linear factors and notice that if  $|\omega| \neq 1$ , then  $(g - \omega)$  is invertible in  $\ell^1(H)$ . Thus since  $A$  and  $B$  are  $\ell^1(H)$ -invariant, we may assume that  $F(g)$  consists of factors  $(g - \omega)$  with  $|\omega| = 1$  only. If we prove that  $F(g)B$  is dense in  $B$ , that will imply that  $A = B$ .

Fix  $\omega$  with  $|\omega| = 1$  and for  $n \in \mathbb{N}$ , let  $x_n = \frac{1}{n} \sum_{k=1}^n \omega^{-k} g^k$ . Note that

$$(3.1) \quad \|x_n\|_p = \left( n \cdot \frac{1}{n^p} \right)^{1/p} = n^{(1-p)/p},$$

consequently  $\lim_{n \rightarrow \infty} \|x_n\|_p = 0$ .

Now pick arbitrary  $b \in B$  and  $\epsilon > 0$ . Since  $\mathbb{C}G^m$  is dense in  $\ell^p(G)^m$ , there exists  $c \in \mathbb{C}G^m$  such that  $\|b - c\|_p < \epsilon/2$ . Then we may choose  $n \in \mathbb{N}$  such that  $\|x_n\|_p < \frac{\epsilon}{2\|c\|_1}$  and we have

$$\begin{aligned} \|x_n b\|_p &= \|x_n(b - c) + x_n c\|_p \leq \|x_n(b - c)\|_p + \|x_n c\|_p \\ &\leq \|x_n\|_1 \|b - c\|_p + \|c\|_1 \|x_n\|_p \quad \text{by (2.2) and (2.3)} \\ &< 1 \cdot \frac{\epsilon}{2} + \|c\|_1 \cdot \frac{\epsilon}{2\|c\|_1} = \epsilon. \end{aligned}$$

Thus  $\|b - (1 - x_n)b\|_p = \|x_n b\|_p < \epsilon$ . Now note that the homomorphism  $\mathbb{C}H \rightarrow \mathbb{C}$  induced by the identity on  $\mathbb{C}$  and sending  $g$  to  $\omega$  has  $(1 - x_n)$  in its kernel, consequently we may write  $1 - x_n = (g - \omega)d$ , where  $d \in \mathbb{C}G$ , and we deduce that  $(1 - x_n)b \in (g - \omega)B$ . Thus  $(g - \omega)B$  is dense in  $B$ . Since the product of operators with dense ranges has dense range by Lemma 2.4, we conclude that  $F(g)B$  is dense in  $B$ .

Therefore we may assume that every element of  $G$  has finite order. Let  $N$  denote the kernel of the action of  $G$  on  $B/A$ . Suppose  $N$  is infinite. Choose an infinite sequence  $\{g_1, g_2, \dots\}$  of distinct elements of  $N$  and let  $x_n = \sum_{k=1}^n \frac{1}{n} g_k \in \mathbb{C}G$ . Let  $b \in B$ , let  $\epsilon > 0$  and follow the argument above. Since  $\|x_n\|_p = n^{(1-p)/p}$ , we see that  $\lim_{n \rightarrow \infty} \|x_n\|_p = 0$ . Also  $\mathbb{C}G^m$  is dense in  $\ell^p(G)^m$ , hence there exists  $c \in \mathbb{C}G^m$  such that  $\|b - c\|_p < \epsilon/2$ . Then we may choose  $n \in \mathbb{N}$  such that  $\|x_n\|_p < \frac{\epsilon}{2\|c\|_1}$  and we have

$$\begin{aligned} \|x_n b\|_p &= \|x_n(b - c) + x_n c\|_p \leq \|x_n(b - c)\|_p + \|x_n c\|_p \\ &\leq \|x_n\|_1 \|b - c\|_p + \|c\|_1 \|x_n\|_p \quad \text{by (2.2) and (2.3)} \\ &< 1 \cdot \frac{\epsilon}{2} + \|c\|_1 \cdot \frac{\epsilon}{2\|c\|_1} = \epsilon. \end{aligned}$$

Thus  $\|b - (1 - x_n)b\|_p = \|x_n b\|_p < \epsilon$ . Since  $(1 - x_n)b \in A$  for all  $n \in \mathbb{N}$ , we see that  $A$  is dense in  $B$  and we conclude that  $A = B$ .

Therefore we may assume that  $N$  is finite, so  $G/N$  is an infinite torsion group, and its action on  $B/A$  tells us that it is also a linear group over  $\mathbb{C}$ . By a theorem of Schur [4, cf. 1.L.4], there is a normal abelian subgroup  $K/N$  of finite index in  $G/N$ . Since a simple  $\mathbb{C}[K/N]$ -module has dimension one over  $\mathbb{C}$ , there is a one-dimensional  $K$ -invariant subspace  $U/A$  of  $B/A$ . Again follow the proof above. Choose an infinite sequence  $\{g_1, g_2, \dots\}$  of distinct elements of  $K/N$ . Then there exist  $\omega_1, \omega_2, \dots \in \mathbb{C}$  with  $|\omega_i| = 1$  such that  $(g_i - \omega_i)U \subseteq A$ . As before for  $n \in \mathbb{N}$ , set  $x_n = \sum_{k=1}^n \frac{1}{n} \omega_k^{-1} g_k$ . Again (cf. (3.1)),  $\|x_n\|_p = n^{(1-p)/p} \rightarrow 0$  as  $n \rightarrow \infty$ .

Now pick arbitrary  $u \in U$  and  $\epsilon > 0$ . Since  $\mathbb{C}G^m$  is dense in  $\ell^p(G)^m$ , there exists  $c \in \mathbb{C}G^m$  such that  $\|u - c\|_p < \epsilon/2$ . Then we may choose  $n \in \mathbb{N}$  such that

$\|x_n\|_p < \frac{\epsilon}{2\|c\|_1}$  and we have

$$\begin{aligned} \|x_n u\|_p &= \|x_n(u - c) + x_n c\|_p \leq \|x_n(u - c)\|_p + \|x_n c\|_p \\ &\leq \|x_n\|_1 \|u - c\|_p + \|c\|_1 \|x_n\|_p \quad \text{by (2.2) and (2.3)} \\ &< 1 \cdot \frac{\epsilon}{2} + \|c\|_1 \cdot \frac{\epsilon}{2\|c\|_1} = \epsilon. \end{aligned}$$

Since  $(1 - x_n)u \in A$  for all  $n$ , we deduce that  $A$  is dense in  $U$ , a contradiction. This completes the proof of Theorem 1.5.  $\square$

*Deduction of Theorem 1.4 from Theorem 1.5.* For Theorem 1.4(i), note that the maps  $d_n^*: \ell^p(G)^{e_n} \rightarrow \ell^p(G)^{e_{n+1}}$  are continuous, because they are given by right multiplication by an  $e_n \times e_{n+1}$  matrix with entries in  $\mathbb{C}G$ . Thus if  $\text{im } d_{n-1}^*$  has finite codimension in  $\ker d_n^*$ , it will be closed by Lemma 2.5. Theorem 1.4(i) now follows from Theorem 1.5. The proof of Theorem 1.4(ii) is almost exactly the same, except we need to deal with right  $G$ -invariant subspaces of  $\ell^p(G)^{e_n}$ .  $\square$

We can also prove results for the corresponding real Banach spaces. For a group  $G$  and  $1 \leq p \in \mathbb{R}$ , let

$$l^p(G) = \left\{ \sum_{x \in G} a_x x \mid a_x \in \mathbb{R} \text{ and } \sum_{x \in G} |a_x|^p < \infty \right\}.$$

This is a real Banach space with respect to the norm  $\|f\|_p := (\sum_{x \in G} |a_x|^p)^{1/p}$ . Also  $G$  acts on the left of  $l^p(G)$  according to the rule  $g \sum_{x \in G} a_x x = \sum_{x \in G} a_x gx$ . Then we have

**Corollary 3.2.** *Let  $G$  be an infinite group, let  $m$  be a non-negative integer, and let  $A \subseteq B$  be closed left  $G$ -invariant subspaces of  $l^p(G)^m$ . Then either  $A = B$  or  $B/A$  has infinite dimension over  $\mathbb{R}$ .*

*Proof.* We can regard  $A$  and  $B$  as closed  $G$ -invariant real subspaces of  $\ell^p(G)^m$ . Set  $X = A + iA$  and  $Y = B + iB$ . Then  $X$  and  $Y$  are closed left  $G$ -invariant complex subspaces of  $\ell^p(G)^m$ . Since either  $Y = X$  or  $Y/X$  has infinite  $\mathbb{C}$ -dimension by Theorem 1.5, we see that either  $B = A$  or  $B/A$  has infinite  $\mathbb{R}$ -dimension and the result follows.  $\square$

#### 4. APPLICATIONS TO FINITELY GENERATED GROUPS

In this section we will use Corollary 3.2 to obtain some new results concerning finitely generated infinite groups. Let  $\mathcal{F}(G)$  denote the set of all real valued functions on  $G$ . This has a left and right  $G$ -action given by  $(gf)(x) = f(g^{-1}x)$  and  $(fg)(x) = f(xg^{-1})$  for  $f \in \mathcal{F}(G)$  and  $g, x \in G$ , respectively. We will view  $l^p(G)$  as  $\{f \in \mathcal{F}(G) \mid \sum_{x \in G} |f(x)|^p < \infty\}$ . To make this identification, we send  $f$  to  $\sum_{x \in G} f(x)x$ . Also  $l^\infty(G) = \{f \in \mathcal{F}(G) \mid \sup_{x \in G} |f(x)| < \infty\}$  with norm  $\|f\|_\infty = \sup_{x \in G} |f(x)|$ . Finally  $\mathbb{R}G$  will denote the functions in  $\mathcal{F}(G)$  with finite support.

Throughout this section,  $p$  will always denote a real number greater than one and  $G$  will be a group with a finite symmetric generating set  $S$  (so  $S = S^{-1}$ ). For a real-valued function  $f$  on  $G$ , we define the  $p$ -th power of the *gradient*, the

$p$ -Dirichlet sum, and the  $p$ -Laplacian of  $g \in G$  by

$$\begin{aligned} |Df(g)|^p &= \sum_{s \in S} |f(g) - f(gs)|^p, \\ I_p(f) &= \sum_{g \in G} |Df(g)|^p, \text{ and} \\ \Delta_p f(g) &= \sum_{s \in S} |f(gs) - f(g)|^{p-2} (f(gs) - f(g)), \end{aligned}$$

respectively. In the case  $1 < p < 2$ , we make the convention that  $|f(gs) - f(g)|^{p-2} (f(gs) - f(g)) = 0$  if  $f(gs) = f(g)$ . We shall say that  $f$  is  $p$ -Dirichlet finite if  $I_p(f) < \infty$ . The set of all  $p$ -Dirichlet finite functions on  $G$  will be denoted by  $D_p(G)$ . A function  $f$  is said to be  $p$ -harmonic if  $\Delta_p f(g) = 0$  for all  $g \in G$ . The set  $HD_p(G)$  will consist of the  $p$ -harmonic functions contained in  $D_p(G)$ . We identify the constant functions on  $G$  with  $\mathbb{R}$ . Observe that  $\mathbb{R}$  is contained in  $HD_p(G)$ . Endowed with the norm

$$\|f\|_{D_p} := (I_p(f) + |f(e)|^p)^{1/p},$$

$D_p(G)$  is a reflexive Banach space, where  $e$  is the identity element of  $G$  and  $f \in D_p(G)$ . For  $X \subseteq D_p(G)$ , let  $\overline{X}_{D_p}$  indicate its closure in the  $D_p$ -norm and let  $B(X)$  denote the bounded functions in  $X$ , that is  $X \cap l^\infty(G)$ ; sometimes we will write  $BX$  for  $B(X)$ . The set  $BD_p(G)$  is closed under the usual operations of scalar multiplication and addition. Also  $BD_p(G)$  is a reflexive Banach space under the norm

$$\|f\|_{BD_p} := (I_p(f))^{1/p} + \|f\|_\infty,$$

where  $f \in BD_p(G)$ . Furthermore,  $\|fh\|_{BD_p} \leq \|f\|_{BD_p} \|h\|_{BD_p}$  for  $f, h \in BD_p(G)$ . Thus  $BD_p(G)$  is an abelian Banach algebra. For  $Y \subseteq BD_p(G)$ , let  $\overline{Y}_{BD_p}$  denote its closure in the  $BD_p$ -norm. Note that if  $f \in BD_p(G)$ , then  $\|f\|_{D_p} \leq \|f\|_{BD_p}$  and that  $B(\overline{\mathbb{R}G}_{D_p}) = B(\overline{l^p(G)}_{D_p})$  is a closed ideal in  $BD_p(G)$ .

Our first application of Corollary 3.2 will be concerned with the cardinality of the  $p$ -harmonic boundary of  $G$ , which we now define. For a more detailed discussion of this boundary, see [9]. Let  $Sp(BD_p(G))$  denote the set of complex-valued characters on  $BD_p(G)$ , that is nonzero  $*$ -homomorphisms from  $BD_p(G)$  to  $\mathbb{C}$ . Then with respect to the weak  $*$ -topology,  $Sp(BD_p(G))$  is a compact Hausdorff space. Given a topological space  $X$ , let  $C(X)$  denote the ring of continuous complex-valued functions on  $X$ . The Gelfand transform defined by  $\hat{f}(\chi) = \chi(f)$  yields a monomorphism of Banach algebras from  $BD_p(G)$  into  $C(Sp(BD_p(G)))$  with dense image. Furthermore the map  $\iota: G \rightarrow Sp(BD_p(G))$  given by  $(\iota(g))(f) = f(g)$  is an injection, and  $\iota(G)$  is an open dense subset of  $Sp(BD_p(G))$ . The  $p$ -Royden boundary of  $G$ , which we shall denote by  $R_p(G)$ , is the compact set  $Sp(BD_p(G)) \setminus \iota(G)$ . The  $p$ -harmonic boundary of  $G$  is the following subset of  $R_p(G)$ :

$$\partial_p(G) := \{\chi \in R_p(G) \mid \hat{f}(\chi) = 0 \text{ for all } f \in B(\overline{\mathbb{R}G}_{D_p})\}.$$

We can now state

**Theorem 4.1.** *Let  $1 < p \in \mathbb{R}$  and let  $G$  be a finitely generated infinite group. Then the cardinality of  $\partial_p(G)$  is either 0, 1 or  $\infty$ .*

*Proof.* Let  $S := \{s_1, \dots, s_d\}$  be a symmetric generating set for  $G$ . We will use the results of [9] with  $G$  being the Cayley graph of  $G$  with respect to the generating set

$S$ ; thus the vertices of this graph are the elements of  $G$ , and  $g_1, g_2 \in G$  are joined by an edge if and only if  $g_1 = g_2 s$  for some  $s \in S$ . If  $1 \in B(\overline{\mathbb{R}G}_{D_p})$ , then  $\partial_p(G) = \emptyset$  by [9, Theorem 2.1 and Proposition 4.2]. Thus we will assume that  $1 \notin B(\overline{\mathbb{R}G}_{D_p})$ . If  $BHD_p(G) = \mathbb{R}$ , then [9, Theorem 4.11] says that  $|\partial_p(G)| = 1$ . Now suppose  $|\partial_p(G)| > 1$ . We will complete the proof of the theorem by showing that if  $|\partial_p(G)|$  is finite, then there exist two closed left  $G$ -invariant subspaces  $A$  and  $B$  of  $l^p(G)^d$  that violate Corollary 3.2. We start by showing how  $D_p(G)$  is related to  $l^p(G)^d$ . Define a continuous linear map  $D_p(G) \rightarrow l^p(G)^d$  by  $\theta(f) = (f(s_1 - 1), \dots, f(s_d - 1))$ . Then  $\ker \theta = \mathbb{R}$  and so  $\theta$  induces an embedding  $\theta': D_p(G)/\mathbb{R} \hookrightarrow l^p(G)^d$ . Clearly  $D_p(G)/\mathbb{R}$  is a Banach space under the norm induced by the norm on  $D_p(G)$ , and  $\theta'$  preserves this norm and also the left  $G$ -action.

We now construct the subspaces of  $l^p(G)^d$  that will give us our contradiction. The Gelfand transform yields a homomorphism of  $BD_p(G)/B(\overline{\mathbb{R}G}_{D_p})$  onto a dense subspace of  $C(\partial_p(G))$ . Now [9, Theorem 4.9] shows that if  $f \in BD_p(G)$ , then  $f \in B(\overline{\mathbb{R}G}_{D_p})$  if and only if  $\hat{f} = 0$  on  $\partial_p(G)$ . This tells us that if  $|\partial_p(G)| < \infty$ , then  $\dim_{\mathbb{R}}(BD_p(G)/B(\overline{\mathbb{R}G}_{D_p})) = |\partial_p(G)|$ . Suppose  $1 < |\partial_p(G)| < \infty$ . Then

$$1 \leq \dim_{\mathbb{R}}(BD_p(G)/(B(\overline{\mathbb{R}G}_{D_p}) \oplus \mathbb{R})) < \infty$$

and we deduce that

$$1 \leq \dim_{\mathbb{R}}((BD_p(G) + \overline{\mathbb{R}G}_{D_p})/\overline{\mathbb{R} \oplus \mathbb{R}G}_{D_p}) < \infty.$$

It follows that  $\theta'((BD_p(G) + \overline{\mathbb{R}G}_{D_p})/\mathbb{R})$  is a subspace of  $l^p(G)^d$  properly containing the finite codimensional closed subspace  $\theta'(\overline{\mathbb{R} \oplus \mathbb{R}G}_{D_p}/\mathbb{R})$ . Hence  $\theta'((BD_p(G) + \overline{\mathbb{R}G}_{D_p})/\mathbb{R})$  is closed in  $l^p(G)^d$ . Thus with  $B = \theta'((BD_p(G) + \overline{\mathbb{R}G}_{D_p})/\mathbb{R})$  and  $A = \theta'(\overline{\mathbb{R} \oplus \mathbb{R}G}_{D_p}/\mathbb{R})$ , we obtain a contradiction from Corollary 3.2. So it is impossible for  $1 < |\partial_p(G)| < \infty$ . Therefore  $|\partial_p(G)|$  is either 0, 1 or  $\infty$ .  $\square$

We will now use Corollary 3.2 to obtain some results about translation invariant linear functionals (which we define below) on  $D_p(G)/\mathbb{R}$ . Let  $X$  be a normed space of functions on  $G$ . For  $f \in X$  and  $h \in G$ , the left translation of  $f$  by  $h$ , denoted by  $f_h$ , is the function  $f_h(g) := f(hg)$ . Assume that if  $f \in X$ , then  $f_h \in X$  for all  $h \in G$ ; that is,  $X$  is left translation invariant. We shall say that  $T$  is a translation invariant left functional (TILF) on  $X$  if  $T(f_h) = T(f)$  for  $f \in X$  and all  $h \in G$ . For the rest of this section translation invariant will mean left translation invariant. A common question to ask is that if  $T$  is a TILF on  $X$ , then is  $T$  continuous? For background about the problem of automatic continuity, see [7, 10, 12, 13]. Define

$$\text{Diff}(X) := \text{linear span}\{f_h - f \mid f \in X, h \in G\}.$$

It is clear that  $\text{Diff}(X)$  is contained in the kernel of any TILF on  $X$ . Observe that  $f \in D_p(G)$  if and only if  $f_h - f \in l^p(G)$  for all  $h \in G$ . By definition we have the following inclusions:

$$\text{Diff}(l^p(G)) \subseteq \text{Diff}(D_p(G)/\mathbb{R}) \subseteq l^p(G) \subseteq D_p(G)/\mathbb{R}.$$

The set  $D_p(G)/\mathbb{R}$  is a Banach space under the norm induced from the norm on  $D_p(G)$ . The norm of  $f \in D_p(G)/\mathbb{R}$  will be indicated by  $\|f\|_{D(p)}$  and the closure of a set  $Y \subseteq D_p(G)/\mathbb{R}$  will be denoted by  $\overline{Y}_{D(p)}$ . We can now state

**Lemma 4.2.**  $\overline{\text{Diff}(D_p(G)/\mathbb{R})}_{D(p)} = \overline{l^p(G)}_{D(p)}.$

*Proof.* Let  $f \in l^p(G)$ . By [13, Lemma 1] there is a sequence  $(f_n)$  in  $\text{Diff}(l^p(G))$  that converges to  $f$  in the  $l^p$ -norm. It now follows from Minkowski's inequality that for  $s \in S$ ,

$$\|(f - f_n)_s - (f - f_n)\|_p^p = \sum_{g \in G} |f(sg) - f_n(sg) - (f(g) - f_n(g))|^p \rightarrow 0$$

as  $n \rightarrow \infty$ . Hence  $f \in \overline{\text{Diff}(l^p(G))}_{D(p)}$  which implies  $l^p(G) \subseteq \overline{\text{Diff}(l^p(G))}_{D(p)}$ , and the result follows.  $\square$

Combining the Hahn-Banach theorem with Lemma 4.2 and the fact  $\overline{l^p(G)}_{D(p)} = D_p(G)/\mathbb{R}$  if and only if  $D_p(G)/\overline{l^p(G)} \oplus \mathbb{R}_{D_p} = 0$ , we obtain the following

**Corollary 4.3.** *Let  $1 < p \in \mathbb{R}$ . Then  $D_p(G)/\overline{l^p(G)} \oplus \mathbb{R}_{D_p} \neq 0$  if and only if there exists a nonzero continuous TILF on  $D_p(G)/\mathbb{R}$ .*

It was shown in [11] that if  $G$  is nonamenable, then the only TILF on  $l^p(G)$  is the zero functional. (Consequently every TILF is automatically continuous!) We now show that this is not true for  $D_p(G)/\mathbb{R}$ .

**Proposition 4.4.** *Let  $1 < p \in \mathbb{R}$  and suppose that  $G$  is a finitely generated non-amenable group. If  $D_p(G)/\overline{l^p(G)} \oplus \mathbb{R}_{D(p)} \neq 0$ , then there is a discontinuous TILF on  $D_p(G)/\mathbb{R}$ .*

*Proof.* It is known that  $l^p(G)$  is closed in  $D_p(G)/\mathbb{R}$  if and only if  $G$  is nonamenable [3, Corollary 1]. Also  $D_p(G)/(l^p(G) \oplus \mathbb{R})$  is infinite dimensional by Corollary 3.2. Let  $B$  be a Hamel basis for  $l^p(G)$  and extend it to a Hamel basis  $H$  for  $D_p(G)/\mathbb{R}$ . Now  $H \setminus B$  corresponds to a Hamel basis of  $D_p(G)/(l^p(G) \oplus \mathbb{R})$ . Select a countable subset  $C := \{f_n \mid n \in \mathbb{N}\}$  from  $H \setminus B$ . Define a linear functional  $T$  on  $D_p(G)/\mathbb{R}$  by  $T(f) = 0$  for  $f \in H \setminus C$  and  $T(f_n) = n\|f_n\|_{D(p)}$  for  $n \in \mathbb{N}$ . By [13, Lemma 1]  $\text{Diff}(l^p(G)) = l^p(G)$ , which implies  $\text{Diff}(D_p(G)/\mathbb{R}) = l^p(G)$ . Thus  $T$  is a TILF on  $D_p(G)/\mathbb{R}$ . However  $\left(\frac{f_n}{n\|f_n\|_{D(p)}}\right) \rightarrow 0$  in  $D_p(G)/\mathbb{R}$  and  $T\left(\frac{f_n}{n\|f_n\|_{D(p)}}\right) = 1$  for all  $n$ . Thus  $T$  is discontinuous on  $D_p(G)/\mathbb{R}$ .  $\square$

The free group on two generators provides an example of a group that satisfies Proposition 4.4, see [8, Corollary 4.3] for details. If  $G$  is an infinite amenable group, then by using an argument similar to the proof of Proposition 4.4, we see that there always exists a discontinuous TILF on  $D_p(G)/\mathbb{R}$  (the key point is that  $D_p(G)/(l^p(G) \oplus \mathbb{R})$  will still be infinite dimensional).

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